FINITE TOPOLOGIES AND BOOLEAN MATRICES

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An nxn topogenous matrix is an nxn transitive reflexive Boolean matrix over the set {0,1}. We use the one-to-one correspondence between the set of all nxn topogenous matrices and the collection of all topologies over a set of order n to investigate properties of finite topologies. This correspondence is obtained in [7] by letting each entry a_{ij} in the matrix A equal one if and only if x_j is an element of the minimal T-open neighborhood about x_i in the topological space (X,T).

Let T, T' and T'' be any three topologies on the finite set $X = \{x_1, x_2, \ldots, x_n\}$; let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be the nxn topogenous matrices corresponding respectively to T, T' and T''; and let $N = \{1,2,\ldots,n\}$. We show that $T \subseteq T'$ if and only if $b_{ij} \leq a_{ij}$ for all $i,j \in N$; we show that $c_{ij} = a_{ij}b_{ij}$ for all $i,j \in N$ if and only if T'' is the smallest (least number of open sets) topology on X such that $T \cup T' \subseteq T''$; and we show that $T'' = T \cap T'$ if and only if $C = (A + B)^{n-1}$ under Boolean matrix operations where $1 + 1 = 1$.

Let M be the group (under Boolean matrix multipli-

cation) of all nxn permutation matrices and for any m M let m' be the transpose of m. We define the following terms: $[A] = \{D \mid D = m'Am,\ m \in M\}$, $H[A] = \{m \mid A = m'Am,\ m \in M\}$, and [T] be the set of all topologies homeomorphic to T in X. Also for any finite set V, let o(V) be the number of elements in V. We show that H[A] is a subgroup of M, that $o([T]) = o([A]) = n!/o(H[A])$, and that if $o([T]) = n!$ then T is T_o.

Let f(n) and p(n) be the number of topologies and the number of closed-open topologies respectively on X. We give enumeration formulas for f(n) and p(n). We show that $f(n) \leq (n - 1 + 3^{n-1})f(n-1)$, that $p(n) \leq n!$, and that $\lim_{n\to\infty} p(n)/f(n) = 0$.

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Chapter 1

INTRODUCTION

H. Sharp [10] and V. Krishnamurthy [6] have shown that a topology on a finite set can be represented by a suitable matrix. These matrices have been found to be interesting tools in investigating the structure of finite topological spaces and in answering related combinatorial questions.

One such suitable matrix is the topogenous matrix, an nxn transitive reflexive matrix over the set {0,1}. A one-to-one correspondence between the set of all nxn topogenous matrices $A = (a_{ij})$ and the collection of all topologies T on the set $X = \{x_1, x_2, \ldots, x_n\}$ is obtained by letting each entry a_{ij} in the matrix A equal one if and only if x_j is an element of the minimal T-open neighborhood about x_i in X. Topogenous matrices coincide with the matrices used by Krishnamurthy [6], Rayburn [8] and Shiraki [11], and with the transpose of the matrices used by Sharp [10].

In this paper we will attempt to present a survey of the major work on finite topology and offer at least partial solutions to some questions previously unanswered. A primary purpose of this paper will be to provide a consistant set of mathematical tools to facilitate future research in this area. Where space permits, we will demonstrate the use of these tools in solving problems related to topologies on a finite set.

REVIEW OF LITERATURE

Solutions to problems concerning finite topological spaces are of importance in many fields of mathematics because of the one-to-one correspondence between topologies on a given set and quasi-orders on the same set. This relation, along with the more commonly known one-to-one correspondence between T_o topologies and partial orders, is noted by Sharp [10], Shiraki [11] and others.

Butler and Markowsky [1] prove enumeration formulas showing the relationships among the number of distinct topologies, T_o topologies, connected topologies and T_o-connected topologies on a set of order n (we display these formulas in theorems 5.6 and 5.7). In a similar fashion they also work with the number of homeomorphism classes of topologies, et cetera, on a set of order n.

By developing the relationship between topologies and quasi-orders on a set X, Kleitman and Rothschild [5] prove that the number of distinct topologies on a set of order n is asymptotic to and bounded below by $2^{n^2/4}$.

Krishnamurthy [6] shows a development of the previously mentioned one-to-one correspondence between topologies and topogenous matrices, and uses this relationship to prove that an upper bound on the number of distinct topologies on a set of order n is 2^{n^2-n}.

In [8], Rayburn is primarily concerned with finite closed-open topologies (i. e. finite Borel fields). However an observation of his (see remark 3.1) forms the basis for the work in chapter three. Among the more interesting results proven in [8] are: that the collection of all closed-open topologies on a finite set of order n forms a complemented lattice, that the number of elements in this lattice is less than or equal to $2^{\frac{1}{2}n(n-1)}$, and that a finite topology is closed-open if and only if its corresponding topogenous matrix is symmetric with respect to the main diagonal.

Two of the more extensive papers are by Sharp [10] and Shiraki [11]. Both of these papers contain many results which are used elsewhere in this thesis. The results proven in [10] include: that a reflexive Boolean matrix is topogenous if and only if it is idempotent; that a topogenous matrix corresponds to a topology if and only if the transpose of the matrix corresponds to the dual of the topology, that a topology on a set of order n is not connected if and only if the corresponding nxn topogenous matrix and its transpose both contain the same k x (n - k) zero submatrix for some k, 0 < k < n; that a topology is T_0

if and only if its corresponding topogenous matrix is anti-symmetric; and that on a set of order n where n is greater than one, the number of T_0 topologies is odd, the number of connected topologies is odd and the number of T_0 connected topologies is even.

Many of the results shown by Shiraki [11] are quite similar to those shown by Sharp [10]. Among other work, Shiraki [11] proves results on component spaces, product spaces and dual spaces. He also works with the concept of degree of connection of a T_0 topology.

PRELIMINARY DEFINITIONS

One of the first results given in this paper will show a one-to-one correspondence between the collection of all topologies on a set of order n and the collection of all nxn reflexive transitive Boolean matrices, and much of the subsequent work will be written in terms of these matrices. Hence it will be to our advantage to first define and discuss Boolean matrices in general.

<u>Definition</u> <u>1.1</u> Let $\{0,1\}$ be a set with Boolean operations + and # defined by $1 + 1 = 1$, $0 + 1 = 1$, $1 + 0 = 1$, $0 + 0 = 0$, $1\#1 = 1$, $1\#0 = 0$, $0\#1 = 0$ and $0\#0 = 0$. For simplicity we will indicate the operation # by juxtaposition.

<u>Definition 1.2</u> Let any nxn matrix containing only zeros or ones be called a Boolean matrix. For any nxn Boolean matrices (a_{ij}), (b_{ij}) and (c_{ij}), and for $i,j \in \{1,2,\ldots,n\}$, we define:

a). $(a_{ij}) = (b_{ij})$ if and only if $a_{ij} = b_{ij}$ for each i and j,

b) $(a_{ij}) \leq (b_{ij})$ if and only if $a_{ij} \leq b_{ij}$ for each i and j,

c) $(a_{ij}) < (b_{ij})$ if and only if $(a_{ij}) \leq (b_{ij})$ and there exists i and j such that $a_{ij} < b_{ij}$,

d) $(c_{ij}) = (a_{ij}) + (b_{ij})$ if and only if $c_{ij} = a_{ij} + b_{ij}$ for each i and j (where $a_{ij} + b_{ij}$ is taken to be Boolean addition), and

e) $(c_{ij}) = (a_{ij})(b_{ij})$ if and only if $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$ for each i and j. We will denote Boolean matrix multiplication of k factors of (a_{ij}) by $(a_{ij})^k$.

We will be discussing Boolean matrices possessing certain special properties and in anticipation we define the following terminology:

Definition 1.3 Let (a_{ij}) be any nxn Boolean matrix and write (a_{ij}) is:

a) reflexive when $a_{ii} = 1$ for each i,

b) symmetric when $a_{ij} = a_{ji}$ for each i and j,

c) antisymmetric when (a_{ij}) is reflexive and for each i and j $(i \neq j)$, $a_{ij} = 1$ implies $a_{ji} = 0$,

d) transitive when for each i, j and k, $a_{ik}a_{kj} = 1$ implies $a_{ij} = 1$,

e) idempotent when $(a_{ij}) = (a_{ij})^2$.

Further discussion of Boolean matrices may be found in papers by R. L. Davis [2] and by O. Feichtinger and B. McAllister [3].

Chapter 2

ELEMENTARY PROPERTIES

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let T be any topology on X. Since X is finite, each x_i in X has a unique minimal open neighborhood U_i in (X,T) and the collection of sets $\{U_1, U_2, \ldots, U_n\}$ is an open neighborhood basis of the topological space (X,T). By using these minimal open neighborhoods, we shall now define a special type of Boolean matrices which will be used throughout our investigation of topologies on a finite set.

Definition 2.1 (M. Shiraki) For a finite space (X,T), $X = \{x_1, x_2, \ldots, x_n\}$, let U_i be the minimal open neighborhood about $x_i \in X$ and let an nxn matrix $A = (a_{ij})$ be defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in U_i \\ 0 & \text{otherwise, for } i,j \in \{1,2,\ldots,n\}. \end{cases}$$

Then the matrix A is said to be a topogenous matrix of the topological space (X,T).

A topogenous matrix has the following important properties:

<u>Theorem 2.2</u> (V. Krishnamurthy) Let A be the topogenous matrix of the finite topological space (X,T). Then the matrix A has the following three properties: A is a Boolean matrix, A is reflexive, and A is transitive. Conversely, if a matrix A has these three properties then A uniquely induces a topology on X and A is a topogenous matrix.

Note that we may now use an nxn topogenous matrix to construct a topology T on $X = \{x_1, x_2, \ldots, x_n\}$ by constructing a collection of subsets of X, setting $x_j \in U_i$ if $a_{ij} = 1$ and otherwise $x_j \notin U_i$ for each $i,j \in \{1,2,\ldots,n\}$. It follows from definition 2.1 and theorem 2.2 that these sets U_i will be the minimal open neighborhoods in (X,T).

It is important to note that any nxn transitive reflexive Boolean matrix is an nxn topogenous matrix, since the rows of the Boolean matrix used by Sharp [10] to correspond to a topological space were defined by the closures of the singleton sets in X. However, Sharp's matrices are also nxn transitive reflexive Boolean matrices and further the columns of a Sharp matrix correspond to the minimal open nieghborhoods of the defining topology (Sharp [10], theorem 1). Thus as Krishnamurthy noted in [7], the transpose of a Sharp matrix for a given topological space is the topogenous matrix for that same topological space and, with care, we may apply theorems in Sharp's paper [10] to topogenous matrices.

Before discussing more useful theorems, we need to develop some equivalent definitions of a topogenous matrix.

Lemma 2.3 Let $A = (a_{ij})$ be any reflexive nxn Boolean matrix. Then the following statements are equivalent:

a) A is a topogenous matrix.

b) A is a transitive matrix.

c) If $a_{ij} = 0$ then $\sum_{k=1}^{n} a_{ik}a_{kj} = 0$ for all $i,j \in \{1,2,\ldots,n\}$.

d) A is an idempotent matrix.

Proof:

(a) $\Longleftrightarrow$ (b): By theorem 2.2.

(b) $\Rightarrow$ (d): Choose $B = (b_{ij})$ such that $A^2 = B$. Note that for each i and j, $b_{ij} = a_{i1}a_{1j} + \ldots + a_{ii}a_{ij} + \ldots + a_{in}a_{nj}$ and thus $b_{ij} \geq a_{ii}a_{ij}$. Since A is reflexive, $a_{ii} = 1$ for each i and thus for each i and j, $b_{ij} \geq a_{ij}$. Hence by definition 1.2 we have that $A^2 \geq A$.

Now if $A^2 > A$ then by definition 1.2, there exists

$i,j \in \{1,2,\ldots,n\}$ such that $b_{ij} > a_{ij}$ and hence for this choice of i and j, $b_{ij} = 1$ and $a_{ij} = 0$. Thus $b_{ji} = a_{i1}a_{1j} + a_{i2}a_{2j} + \ldots + a_{in}a_{nj} = 1$ implies there exists $k \in \{1,2,\ldots,n\}$ such that $a_{ik}a_{kj} = 1$, and hence by (b), that $a_{ij} = 1$. Thus we have reached the contradiction that a_{ij} must be both zero and one. Hence we are forced to the required conclusion that (b) implies (d).

(d) $\Rightarrow$ (c): Assume that $A = A^2$. Then by definition 1.2, $a_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} + \ldots + a_{in}a_{nj}$. Hence if $a_{ij} = 0$ then $a_{i1}a_{1j} + a_{i2}a_{2j} + \ldots + a_{in}a_{nj} = 0$ and (d) implies (c) as required.

(c) $\Rightarrow$ (b): If A is not transitive then there exists $i,j,k \in \{1,2,\ldots,n\}$ such that $a_{ik}a_{kj} = 1$ and $a_{ij} = 0$. But by (c) we have that $a_{ij} = 0$ implies that $a_{i1}a_{1j} + a_{i2}a_{2j} + \ldots + a_{in}a_{nj} = 0$, and in particular $a_{ik}a_{kj} = 0$. Thus we reach the contradiction that $a_{ik}a_{kj}$ must be both zero and one. Hence (c) implies (b) as required, completing the proof of the lemma.

To simplify notation in later work, we make the following definition concerning row and column tuples:

Definition 2.4 For any two Boolean matrices A and B such that A and B are both nxn, let a_{i*} and b_{j*} (a_{*i} and

b_{*j}) be the i^{th} and j^{th} row tuples (column tuples) of A and B respectively, and write:

a) $a_{i*} \leqq b_{j*}$ ($a_{*i} \leqq b_{*j}$) if and only if $a_{ik} \leqq b_{jk}$ ($a_{ki} \leqq b_{kj}$) for $k = 1,2,\ldots,n$.

b) $a_{i*} < b_{j*}$ ($a_{*i} < b_{*j}$) if and only if $a_{i*} \leqq b_{j*}$ ($a_{*i} \leqq b_{*j}$) and there exists $k \in \{1,2,\ldots,n\}$ such that $a_{ik} < b_{jk}$ ($a_{ki} < b_{kj}$).

Using the above notation we can now adapt the following theorem and corollary from [10]:

<u>Theorem</u> <u>2.5</u> (Sharp) If (a_{ij}) is an nxn topogenous matrix then for any given $i,j \in \{1,2,\ldots,n\}$ the following are equivalent:

a) $a_{ij} = 1$

b) $a_{i*} \geqq a_{j*}$

c) $a_{*i} \leqq a_{*j}$.

<u>Corollary</u> <u>2.6</u> (Sharp) If (a_{ij}) is an nxn topogenous matrix then for any given $i,j \in \{1,2,\ldots,n\}$ the following are equivalent:

a) $a_{ij} = a_{ji}$

b) $a_{j*} = a_{i*}$

c) $a_{*i} = a_{*j}$.

The following theorem shows the correspondence between an order relation on the set of all topologies on $X = \{x_1, x_2, \ldots, x_n\}$ and an order relation on the set of all nxn topogenous matrices.

<u>Theorem</u> <u>2.7</u> Let T and T^* be any two topologies on a set $X = \{x_1, x_2, \ldots, x_n\}$ and let $A = (a_{ij})$ and $B = (b_{ij})$ be their corresponding nxn topogenous matrices. Then $T \subseteq T^*$ if and only if $A \geq B$.

<u>Proof</u>: ($\Rightarrow$) If $T \subseteq T^*$ then for each pair of minimal open neighborhoods U_i and U_i^* of $x_i \in X$ in T and T^* respectively, $U_i \supseteq U_i^*$. Thus it follows by definition 2.1 that for each i and j, $a_{ij} \geq b_{ij}$ and $A \geq B$.

Note that the above argument can be reversed, completing the theorem.

As final results we note the following theorem and corollary on the correspondence between T_o topologies and topogenous matrices.

Theorem 2.8 (Sharp) If T is any topology on a finite set and A is its corresponding topogenous matrix then T is T_o if and only if A is antisymmetric.

Corollary 2.9 If T is any topology on a set of order n and (a_{ij}) is its corresponding nxn topogenous matrix then T is T_o if and only if $a_{i*} \neq a_{j*}$ for all $i \neq j$, $i,j \in \{1,2,\ldots,n\}$.

Proof: By definition 1.3 and corollary 2.6, A is antisymmetric if and only if $a_{i*} \neq a_{j*}$ for all $i \neq j$, $i,j \in \{1,2,\ldots,n\}$, and hence the corollary follows from theorem 2.8.

Chapter 3

CORRESPONDING OPERATIONS

<u>Remark</u> <u>3.1</u> Given two topologies T and T^* on a finite set X, and their corresponding topogenous matrices A and B, M. Rayburn [8] has made the observation that the element-wise direct product of A and B is a topogenous matrix corresponding to the smallest (least number of open sets) topology containing the set $T \bigcup T^*$. He also noted that the smallest (least number of nonzero elements) topogenous matrix containing the Boolean matrix $A + B$ corresponds to the topology $T \bigcap T^*$.

We will show that Rayburn's observations are correct and further that $(A + B)^{n-1}$ is the smallest topogenous matrix containing the Boolean matrix $A + B$.

Since the union of two topologies on a set X is not necessarily a topology on X, we will begin by working with a general collection of subsets of X. Let $X = \{x_1, x_2, \ldots, x_n\}$ and let S be a collection of subsets of X such that $X \varepsilon S$. Define S_i by $S_i = \bigcap\{V | V\varepsilon S \text{ and } x_i \varepsilon V\}$. It is a standard exercise (Kelley, [4], chapter 1, exercise B) that the collection of sets $\{S_i \mid i = 1,2,\ldots,n\}$ forms the unique minimal open neighborhood basis for some topology T on X.

<u>Lemma</u> <u>3.2</u> T as constructed above is the smallest topology on X such that $S \subseteq T$.

<u>Proof</u>: Let $T^* = \{R \mid R$ is a topology on X and $S \subseteq R\}$. It is obvious that $S \subseteq T$. Hence T^* is not an intersection over the empty set and $T^* \subseteq T$.

For $i = 1,2,\dots,n$ let U_i^* be the minimal open neighborhood about $x_i \in X$ in T^*. Thus for each i, $S \subseteq T^*$ implies $U_i^* \subseteq S_i$, and since T is the topology formed by the arbitrary union of elements of $\{S_i \mid i = 1,2,\dots,n\}$, we have that $T \subseteq T^*$. Hence $T = T^*$ as required.

<u>Definition</u> <u>3.3</u> For any two nxn Boolean matrices $A = (a_{ij})$ and $B = (b_{ij})$ let $C = (c_{ij})$ be the nxn Boolean matrix defined by $c_{ij} = a_{ij}b_{ij}$ for i,j $1,2,\dots,n$ and write $C = A \wedge B$.

Now that we have defined the elementwise direct product between matrices, we are ready to verify Rayburn's observation (see remark 3.1) concerning direct products of topogenous matrices.

<u>Theorem</u> <u>3.4</u> Let T_1 and T_2 be any two topologies on a

finite set $X = \{x_1, x_2, \ldots, x_n\}$ and let $A = (a_{ij})$ and $B = (b_{ij})$ be their corresponding nxn topogenous matrices. Then:

a) the matrix $C = A \wedge B$ is a topogenous matrix, and

b) the topology T corresponding to the topogenous matrix C is the smallest topology on X which contains the set $S = T_1 \cup T_2$.

Proof (a): By definition 3.3 we have the following:

1) for all i, $c_{ii} = a_{ii}b_{ii} = 1$ and

2) if $c_{ik} = 1 = c_{kj}$ then $c_{ik} = a_{ik}b_{ik} = 1$ and $c_{kj} = a_{kj}b_{kj} = 1$.

Thus if $c_{ik}c_{kj} = 1$ then $a_{ik}a_{kj} = 1$ and $b_{ik}b_{kj} = 1$. Hence $a_{ij} = 1$ and $b_{ij} = 1$, and $c_{ij} = a_{ij}b_{ij} = 1$. Therefore by lemma 2.3 C is a topogenous matrix.

(b): Let U_i, U_i', U_i'' and S_i be the minimal open neighborhoods about $x_i \in X$ in T, T_1, T_2 and S respectively. Note that if we show $U_i = S_i$ for all i, then by lemma 3.2 we are done.

Note that $S = T_1 \cup T_2$ implies that for each $x_i \in X$, $\{V |\ V \in S \text{ and } x_i \in V\} = \{U |\ U \in T_1 \text{ and } x_i \in U\} \cup \{U |\ U \in T_2 \text{ and } x_i \in U\}$. Hence if $S_i = \bigcap\{V |\ V \in S \text{ and } x_i \in V\}$ then
$S_i = (\bigcap\{U |\ U \in T_1 \text{ and } x_i \in U\}) \cap (\bigcap\{U |\ U \in T_2 \text{ and } x_i \in U\})$.

But, $U_i' = \bigcap\{U \mid U \varepsilon T_1 \text{ and } x_i \varepsilon U\}$ and $U_i'' = \bigcap\{U \mid U \varepsilon T_2 \text{ and } x_i \varepsilon U\}$, and thus $S_i = U_i' \cap U_i''$.

Now we show that $U_i = U_i' \cap U_i''$. By the definition of the matrix C, $c_{ij} = a_{ij} b_{ij}$ for all i andj. Thus by the definition of a topogenous matrix, $x_j \varepsilon U_i$ if and only if $c_{ij} = 1 = a_{ij} b_{ij}$ if and only if $x_j \varepsilon U_i'$ and $x_j \varepsilon U_i''$, which holds if and only if $x_j \varepsilon U_i' \cap U_i''$. So $U_i = U_i' \cap U_i''$. Hence $U_i = S_i$ and thus by lemma 3.2, the topology T is the smallest topology on X which contains $S = T_1 \cup T_2$, as required.

We need now to investigate the topogenous matrix operation which corresponds to taking the intersection of two topologies on a set X. Consider two topologies T_1 and T_2 and their intersection $T = T_1 \cap T_2$ on a finite set X. If U_i, U_i' and U_i'' are the minimal open neighborhoods about $x_i \varepsilon X$ in T, T_1 and T_2 respectively, then $T = T_1 \cap T_2$ implies that $U_i \supseteq U_i' \cup U_i''$. Note that unfortunately, it is not necessarily true that $U_i = U_i' \cup U_i''$.

For the same three topologies T, T_1 and T_2 let A, B and C be their respective topogenous matrices and let $D = B + C$ where $d_{ij} = b_{ij} + c_{ij}$ for each i and j (note we are still using Boolean operations). Observe that: $x_j \varepsilon U_i' \cup U_i''$ if and only if $b_{ij} = 1$ or $c_{ij} = 1$, which holds if and only if $d_{ij} = b_{ij} + c_{ij} = 1$. It follows that $A \geq B + C = D$. Computation of a few examples will show that

D is not necessarily a topogenous matrix, as might be expected since $U_i' \cup U_i''$ is not necessarily a minimal open neighborhood in $T_1 \quad T_2$.

Thus the major part of the work in this section will be to investigate nxn reflexive Boolean matrices, in order to generate a topogenous matrix from a Boolean matrix that is not necessarily topogenous. We will show that if D is an nxn reflexive Boolean matrix, then D^{n-1} is topogenous and we will then prove a stronger form of Rayburn's second observation (see remark 3.1).

In the following work, unless specifically stated otherwise, all matrices will be nxn reflexive Boolean matrices, and not necessarily topogenous. Operations and relations will be as defined in definitions 1.1, 1.2 and 1.3, and in addition we make the following two definitions:

Definition 3.5 Let A and B be any nxn reflexive Boolean matrices and define $A \bigvee B$ by $A \bigvee B = (A + B)^{n-1}$.

Definition 3.6 Let a_{i*} be the i^{th} row tuple of the Boolean matrix A. Define a_{i*}^{m} to be the i^{th} row tuple of A^m where m is any positive integer.

Lemma 3.7 For any nxn reflexive Boolean matrix A and for any positive integer m, $A^{m+1} \geq A^m$.

Proof: Let m be a positive integer, and let $B = (b_{ij})$ and $C = (c_{ij})$ be such that for the nxn reflexive Boolean matrix $A = (a_{ij})$, $B = A^m$ and $C = A^{m+1} = BA$. For any c_{ij} in C, then:

$c_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{ij}a_{jj} + \ldots + b_{in}a_{nj} \geqq b_{ij}a_{jj}$.
Note that since A is reflexive, $a_{jj} = 1$ and hence $c_{ij} \geqq b_{ij}$. Thus by definition 1.2, $C \geqq B$ and $A^{m+1} \geqq A^m$ as required.

According to definitions 2.4 and 3.6, we also have the following corollary to lemma 3.7:

Corollary 3.8 Let a_{i*} be the i^{th} row tuple of the nxn reflexive Boolean matrix A. Then for any positive integer m, $a^{m+1}_{i*} \geqq a^m_{i*}$.

Lemma 3.9 Let a_{i*} be the i^{th} row tuple of the nxn reflexive Boolean matrix A. If there exists $m > 0$ such that $a^m_{i*} = a^{m+1}_{i*}$ then for any positive integer r greater than m, $a^m_{i*} = a^r_{i*}$.

Proof (by induction on $r > m$): Let m be a positive integer. Let $B = A^m$, $C = A^{m+1}$, and $D = A^{m+2}$. Then by lemma 3.7, we have that $A^m \leqq A^{m+1} \leqq A^{m+2}$, and hence $B \leqq C \leqq D$. It follows from definition 1.2 that $b_{ij} \leqq c_{ij} \leqq d_{ij}$ for $i,j \in \{1,2,\ldots,n\}$.

Now assume that $a_{i*}^{m} = a_{i*}^{m+1}$ and hence $b_{ij} = c_{ij}$ for $j = 1,2,\ldots,n$. Then $d_{ij} = c_{i1}a_{1j} + c_{i2}a_{2j} +\ldots+ c_{in}a_{nj}$ and thus $d_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j} +\ldots+ b_{in}a_{nj}$. But $c_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j} +\ldots+ b_{in}a_{nj}$ and hence $d_{ij} = c_{ij} = b_{ij}$ for $j = 1,2,\ldots,n$. Thus $a_{i*}^{m} = a_{i*}^{m+1} = a_{i*}^{m+2}$ and the lemma is correct for $r = m + 2$. So assume that the lemma is correct up to some integer $r > m + 2$. That is assume $a_{i*}^{m} = a_{i*}^{m+1}$ implies that $a_{i*}^{m} = a_{i*}^{m+1} =\ldots= a_{i*}^{r-1}$.

We will now show that $a_{i*}^{m} = a_{i*}^{m+1} =\ldots= a_{i*}^{r-1}$ implies that $a_{i*}^{r-1} = a_{i*}^{r}$. If we let $B = A^{r-2}$, $C = A^{r-1}$ and $D = A^{r}$, then the argument follows exactly as before, completing the proof as required.

<u>Corollary 3.10</u> Let r and m be positive integers with $r > m$ and let A be any nxn reflexive Boolean matrix. If $A^{m} = A^{m+1}$ then $A^{m} = A^{r}$ for all $r > m$ and A^{m} is idempotent.

<u>Proof</u>: If for some $m > 0$, $A^{m} = A^{m+1}$ then $a_{i*}^{m} = a_{i*}^{m+1}$ for $i = 1,2,\ldots,n$ and it follows from lemma 3.9 that $a_{i*}^{m} = a_{i*}^{r}$ for all $r > m$.

Now choose $r = 2m$. If $A^{m} = A^{m+1}$ then we have that $A^{m} = A^{2m} = (A^{m})^{2}$ and hence by definition 1.3, A^{m} is idempotent, completing the proof.

<u>Theorem 3.11</u> If A is an nxn reflexive Boolean matrix then $A^{n-1} = A^{n}$ and A^{n-1} is topogenous.

Proof: For each $m < n$, let a^m_{i*} be the i^{th} row of A^m. If for each $i = 1,2,\ldots,n$ there exists an m such that $m < n$ and $a^m_{i*} = a^{m+1}_{i*}$, then by lemma 3.9, $a^m_{i*} = a^{n-1}_{i*} = a^n_{i*}$ (note that m is dependent on i and may be different for distinct choices of i). But then for all i, we have that $a^{n-1}_{i*} = a^n_{i*}$ and hence $A^{n-1} = A^n$.

Thus assume that there exists $i \varepsilon \{1,2,\ldots,n\}$ such that $a_{i*} < a^2_{i*} <\ldots< a^n_{i*}$. Let $(b_{ij}) = A^2$ and note that for each b_{ij} in a^2_{i*}, $b_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} +\ldots+ a_{in}a_{nj}$. Hence if $a_{ik} = 0$ for all $k \neq i$, then $b_{ij} = 0$ for all $j \neq i$ and thus $a_{i*} = a^2_{i*}$. Since we are assuming $a_{i*} < a^2_{i*}$ then we are forced to also assume that there exists $j \varepsilon \{1,2,\ldots,n\}$ such that $j \neq i$ and $a_{ij} = 1$. But $a_{ii} = 1$ and hence we are assuming that a_{i*} contains at least two non-zero elements. Further if a^k_{i*} and a^{k+1}_{i*} (for $1 \leq k < n$) contain the same number of non-zero elements and $a^k_{i*} \leq a^{k+1}_{i*}$ then $a^k_{i*} = a^{k+1}_{i*}$. Hence we are forced to assume that a^2_{i*} contains 3 non-zero elements, a^k_{i*} $(1 \leq k < n)$ contains $k + 1$ non-zero elements and ultimately that a^n_{i*} contains $n + 1$ non-zero elements. But since a^n_{i*} is an n-tuple and contains only n elements, our assumption that $a_{i*} < a^2_{i*} <\ldots< a^n_{i*}$ was false, and hence $A^{n-1} = A^n$.

If A is reflexive, then by lemma 3.7, $A^{n-1} \geq A$ and hence A^{n-1} is reflexive. If $A^{n-1} = A^n$ then by corollary 3.10, A^{n-1} is idempotent. Hence if A is reflexive and $A^{n-1} = A^n$, then by lemma 2.3, A^{n-1} is topogenous, completing the proof as required.

<u>Lemma</u> <u>3.12</u> Let A and B be any nxn Boolean matrices and let m be any positive integer. If $A \leq B$ then $A^m \leq B^m$.

<u>Proof</u>: If $m = 1$ then we are done, so assume that $m \geq 2$. If $A \leq B$ then by multiplying on the left by A, we have: $A^2 \leq AB$ and hence $A^2 \leq B^2$. Thus by induction on m, we have that $A^m \leq A^{m-1}B \leq A^{m-2}B^2 \leq \ldots \leq B^m$ and hence $A^m \leq B^m$ as required.

Recalling that by definition 3.5, $B \vee C = (B+C)^{n-1}$ where B and C are nxn Boolean matrices, we will now prove our stronger version of Rayburn's second observation (see remark 3.1).

<u>Theorem</u> <u>3.13</u> Let T, T_1 and T_2 be topologies of a set $X = \{x_1, x_2, \ldots, x_n\}$ such that $T = T_1 \cap T_2$ and let A, B and C be their respective nxn topogenous matrices. Then $A = B \vee C$.

<u>Proof</u>: Since $T = T_1 \cap T_2$ we note that T is a subset of T_1 and T is a subset of T_2. Hence by lemma 2.7, $A \geq B$ and $A \geq C$. Thus $A \geq B + C$ and so $A^{n-1} \geq (B + C)^{n-1}$ by lemma 3.12. But A is topogenous and so by lemma 2.3 A is idempotent. Thus by corollary 3.10 $A = A^{n-1}$ and hence

$A \geqq (B + C)^{n-1} = B \bigvee C$.

Since B and C are topogenous, B and C are both reflexive, and so B + C is reflexive. Hence by theorem 3.11, $B \bigvee C$ is a topogenous matrix and thus there exists a topology T^* on X corresponding to the topogenous matrix $B \bigvee C$. We note that $B \leqq B + C$ and $C \leqq B + C$, and thus by lemma 3.7 $B \leqq B \bigvee C$ and $C \leqq B \bigvee C$. Hence by theorem 2.7 $T^* \subseteq T_1$ and $T^* \subseteq T_2$. Thus T^* is a subset of T and hence, again by theorem 2.7, $A \leqq B \bigvee C$. Thus $A = B \bigvee C$ as required.

As final results in this section we note the following two corollaries to theorem 3.13. The proof of the first corollary is contained within the proof of theorem 3.13.

<u>Corollary 3.14</u> If B and C are nxn topogenous matrices then $B \bigvee C$ is an nxn topogenous matrix.

The next corollary follows from theorem 2.7 and the preceeding corollary and theorem.(3.13 and 3.14).

<u>Corollary 3.15</u> If T, T_1 and T_2 are topologies on a set of order n, and A, B and C are their respective nxn topogenous matrices, then $T = T_1 \cap T_2$ if and only if $A = B \bigvee C$.

Chapter 4

HOMEOMORPHISM CLASSES

In this section we will investigate homeomorphism classes of topologies on a finite set via topogenous and permutation matrices. We make the following definitions to set up the basic framework for this investigation.

<u>Definition</u> <u>4.1</u> Let $\mathcal{A}$ be the set of all nxn topogenous matrices.

<u>Definition</u> <u>4.2</u> (Shiraki) Define the permutation matrix m which corresponds to the permutation:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n) \end{pmatrix}$$

by $m = (\delta_{i\,\rho(i)})$ where $\delta_{i\,\rho(i)}$ is the Kronecker delta. Let m' be the transpose of m and let M be the group (under Boolean matrix multiplication) of all nxn permutation matrices.

<u>Definition</u> <u>4.3</u> (Shiraki) Given two nxn Boolean matrices A and B, write A is equivalent to B if there exists $m \in M$

such that $B = m'Am$.

Definition 4.4 For any topology T on a finite set X and for any $A \varepsilon \mathcal{A}$, let $[T]$ be the set of all topologies on X homeomorphic to T and let $[A]$ be the set of all matrices in $\mathcal{A}$ equivalent to A.

Definition 4.5 For any finite set S, let $o(S)$ be the number of elements in S.

The corollary to the next result will show that $o([T]) = o([A])$ for a given topology T and its corresponding $A \varepsilon \mathcal{A}$ on a set of order n.

Theorem 4.6 (Shiraki) Let T and T^* be two topologies on a finite set of order n and let $A, B \varepsilon \mathcal{A}$ be their respective topogenous matrices. Then (X,T) and (X,T^*) are homeomorphic if and only if A and B are equivalent.

Corollary 4.7 Let T be a topology on a set of order n and let A be its corresponding topogenous matrix. Then $o([T]) = o([A])$.

The following lemma from [11] will allow us to examine the structure of $[A]$ more closely.

Lemma 4.8 (Shiraki) A matrix which is equivalent to a topogenous matrix is topogenous.

Corollary 4.9 For any $A \varepsilon \mathbb{A}$, $[A] = \{B \mid B = m'Am \text{ for some } m \varepsilon M\}$.

For a given $A \varepsilon \mathbb{A}$ we may not need all of M to generate the particular topogenous matrix equivalence class [A], as the following definition and lemma will show.

Definition 4.10 For any $A \varepsilon \mathbb{A}$, let $H[A] = \{m \varepsilon M \mid m'Am = A\}$.

Lemma 4.11 If $A \varepsilon \mathbb{A}$, then H[A] is a subgroup of M.

Proof: Let e be the identity element of M and note that $e'Ae = A$ for any $A \varepsilon \mathbb{A}$. Thus H[A] is nonempty. If $h_1, h_2 \varepsilon H[A]$, then $h_1'Ah_1 = A$ and $h_2'Ah_2 = A$. Hence $(h_1h_2)'A(h_1h_2) = h_2'h_1'Ah_2h_1 = h_2'Ah_2 = A$, and thus h_1h_2 is an element of H[A], showing that H[A] is a subgroup of M as required.

Lemma 4.12 For any $A \varepsilon \mathbb{A}$ and for any $r,s \varepsilon M$, $r'Ar = s'As$ if and only if r and s are in the same right coset of H[A] in M.

<u>Proof</u>: ($\Rightarrow$) If $r'Ar = s'As$ for $s,r \in M$, then $sr'Ars' = ss'Ass' = A$ and so $(rs')'A(rs') = A$. Thus $(rs') \in H[A]$, and there exists $h \in H[A]$ such that $rs' = h$. Hence $r = hs$, and r and s are in the same right coset of $H[A]$.

($\Leftarrow$) If r and s are in the same right coset of $H[A]$, then there exists $h \in H[A]$ such that $r = sh$. Hence $r'Ar = (hs)'A(hs) = s'h'Ahs$ and since $h'Ah = A$, we have that $r'Ar = s'As$, completing the proof.

It follows from the preceding lemma that for a given $A \in \mathbb{A}$, the number of elements in $[A]$ is equal to the number of distinct right cosets of $H[A]$. We also note that $o(M) = n!$ and that the number of right cosets of $H[A]$ in M is $n!/o(H[A])$. Thus we have proven the following theorem:

<u>Theorem</u> <u>4.13</u> For any $A \in \mathbb{A}$, $o([A]) = n!/o(H[A])$.

By applying corollary 4.7 to theorem 4.13, we have a technique for counting the number of elements in a homeomorphism class of topologies a finite set. We give an example of the use of this technique in the following theorem and corollary.

<u>Theorem</u> <u>4.14</u> Let T be a topology on a set X of order n and let A be its corresponding nxn topogenous matrix.

If $o([A]) = n!$ then T is T_o.

Proof: Assume that $o([A]) = n!$ and T is not T_o. Now $o([A]) = n!$ implies that $o(H[A]) = 1$ by theorem 4.13. Also if T is not T_o then by lemma 2.8, A is not anti-symmetric. Hence there exists i and j such that $a_{ij} = 1 = a_{ji}$ for some a_{ij}, a_{ji} in A, and thus $a_{i*} = a_{j*}$ and $a_{*i} = a_{*j}$ by corollary 2.6. Let the permutation π be $\pi = (i,j)$ and let m_π be the corresponding permutation matrix in M. Then $m_\pi' A m_\pi = A$ and $\{e, m_\pi\} \subseteq H[A]$, where e is the identity element of M. Hence $o(H[A]) \geq 2$. But this contradicts that $o(H[A]) = 1$. Thus our original assumption was false and $o([A]) = n!$ implies T is T_o as required.

In proving lemma 4.14, we also note the following corollary:

Corollary 4.15 If T is not T_o then $o([A]) \leq n!/2$.

Chapter 5

BOUNDS AND ENUMERATIONS

In this section we will investigate various methods of bounding and enumerating the number of elements in various subsets of the collection of all topologies on a given finite set. To facilitate this discussion, we will use the following notation.

<u>Definition</u> <u>5.1</u> On a given set X of order n, let:

a) f(n) be the number of topologies,

b) g(n) be the number of T_o topologies,

c) h(n) be the number of connected topologies,

d) r(n) be the number of T_o-connected topologies,

and

e) p(n) be the number of closed-open topologies.

It should be apparent that an upper bound on $f(n)$ is the number of subsets of the power set of $X = \{x_1, x_2, \ldots, x_n\}$ or in other words:

$$f(n) \leq 2^{2^n}.$$

Krishnamurthy [6] was apparently the first to make a significant improvement on this upper bound by using the one-to-one correspondence between the set of all topologies on a set of order n and the set of all nxn topogenous matrices. Although Krishnamurthy was not able to use the full power of this correspondence, he was able to show that:

$$f(n) \leq 2^{n(n-1)},$$

by merely noting that there were $2^{n(n-1)}$ nxn reflexive Boolean matrices. In the following work we shall construct a better approximation of $f(n)$.

<u>Lemma</u> <u>5.2</u> If $A = (a_{ij})$ is an nxn topogenous matrix and $B = (b_{ij})$ is an $(n-1)x(n-1)$ matrix such that $b_{ij} = a_{ij}$ for all $i,j \in \{1,2,\ldots,n-1\}$, then B is an $(n-1)x(n-1)$ topogenous matrix.

<u>Proof</u>: Since A is topogenous, A is both transitive and reflexive, and since $b_{ii} = a_{ii}$ for $i = 1,2,\ldots,n-1$

B is also reflexive. Thus we need only to show that B is transitive and the result will follow by lemma 2.3.

Assume to the contrary that B is not transitive. Then by definition 1.3, there exists i, j and k such that $b_{ik}b_{kj} = 1$ and $b_{ij} = 0$. But $a_{ik} = b_{ik}$, $a_{kj} = b_{kj}$ and $a_{ij} = b_{ij}$. We are thus forced to the contradiction that A is not transitive. Hence B must be transitive and thus topogenous, completing the proof.

Thus by lemma 5.2, if we take each $(n-1)\times(n-1)$ topogenous matrix and adjoin each possible n^{th} row and column, the set of matrices generated will contain all possible nxn topogenous matrices. If we note that we must have $a_{nn} = 1$, then there are 2^{n-1} ways of choosing an n^{th} row and the same for an n^{th} column. Since $f(n-1) \leq 2^{(n-1)(n-2)}$ and $2^{2(n-1)}2^{(n-1)(n-2)} = 2^{n(n-1)}$, we have shown:

<u>Corollary</u> <u>5.2</u> $f(n) \leq 2^{2n-2}f(n-1) \leq 2^{n(n-1)}$ for $n > 1$.

To produce the new upper bound in corollary 5.2, we have considered every possible choice for an n^{th} row and n^{th} column of a fixed $(n-1)\times(n-1)$ topogenous matrix. But by using corollary 2.6, we can reduce the number of possible choices for this n^{th} row and column. We formalize this concept in the proof of the following theorem:

<u>Theorem 5.3</u> If $n > 1$, then $f(n) \leq (n - 1 + 3^{n-1})f(n)$.

<u>Proof</u>: Fix any $(n-1)x(n-1)$ topogenous matrix and assume that we adjoin an n^{th} row and column such that the nxn matrix $A = (a_{ij})$ thus formed is topogenous.

If in our choice of the n^{th} row and column, there exists elements a_{in} and a_{ni} such that $a_{in}a_{ni} = 1$ for some $i \in \{1,2,\ldots,n-1\}$, then by corollary 2.6, $a_{n*} = a_{i*}$ and $a_{*n} = a_{*i}$. Thus if a single pair of symmetric entries (a_{in}, a_{ni}) in the n^{th} row and column is such that $a_{in} = 1$ and $a_{ni} = 1$, then the rest of the entries in the n^{th} row and column are forced by our choice of the original $(n-1)x(n-1)$ topogenous matrix. Since there are $n - 1$ such pairs (excluding (a_{nn}, a_{nn})) and there are three ways of choosing each pair such that $a_{in}a_{ni} = 0$, there are thus at most $(n - 1 + 3^{n-1})$ ways of choosing an n^{th} row and column to form an nxn topogenous matrix from an $(n-1)x(n-1)$ topogenous matrix. Hence $f(n) \leq (n - 1 + 3^{n-1})$ as required.

If we note that for $n > 2$, $2^{2(n-1)} > n - 1 + 3^{n-1}$, then it follows by corollary 5.2 that for $n > 2$, $2^{n(n-1)} > (n - 1 + 3^{n-1})f(n-1)$.

<u>Remark 5.4</u> In investigating a lower bound on $f(n)$, D. Kleitman and B. Rothschild [5] were able to show that:

$$2^{n^2/4} \leq f(n),$$

for all n > 1 and more importantly that the two quantities were asymptotically equal as n approached infinity.

Corollary 5.5 shows the relationships among f(n) and the various bounds we have discussed. The largest known value of f(n) is f(7), and the values for f(n) are taken from [1].

<u>Corollary 5.5</u> If f(n) is the number of topologies on a set of order n, then:

n	$2^{n^2/4}$	f(n)	†	$2^{2n-2}f(n-1)$	$2^{n(n-1)}$
1	-	1	1	1	1
2	2	4	4	4	4
3	~5	29	44	64	64
4	16	355	870	1,856	4,096
5	~152	6,942	30,175	90,880	1,048,576
6	512	209,527	1,721,616	7,108,608	$\approx 10^9$
7	4,870	9,535,241	154,002,345	858,222,582	$\sim 4 \times 10^{12}$

Where † is replaced by $(n - 1 + 3^{n-1})f(n)$.

We will next consider two theorems developed in [1] which show the numerical relationships among f(n), g(n), h(n) and r(n). These two theorems are included only for completeness, and for further development of them, see [1].

The functions s(n,m) and S(n,m) used in theorem 5.6 and theorem 5.7 denote Stirling numbers of the first and second kind respectively. Recursion relations and generating functions for Stirling numbers may be found in [9] and also in most basic combinatorics texts. For definitions of the other functional symbols used in the following two theorems, see definition 5.1.

<u>Theorem</u> <u>5.6</u> (Butler and Markowsky)

a) $f(n) = \sum_{m=1}^{n} S(n,m)g(n)$,

b) $g(n) = \sum_{m=1}^{n} s(n,m)f(n)$,

c) $h(n) = \sum_{m=1}^{n} S(n,m)r(n)$, and

d) $r(n) = \sum_{m=1}^{n} s(n,m)h(n)$.

The following theorem from [1] was derived via the Bell polynomial, a generalized exponential generating function. For basic properties of the Bell polynomial, the reader is referred to Riordan, [9], sections 2.8 and 4.5.

<u>Theorem</u> <u>5.7</u> (Butler and Markowsky) Let P_n be any partition of n, let k_i be the number of parts of P_n of size i, let $k = k_1 + k_2 + \ldots + k_n$, and let s(k,1) be a Stirling number of the first kind. Then,

$$\text{a)}\quad g(n) = \sum_{P_n} n! \prod_{i=1}^{n} (1/k_i!) \left(\frac{r(i)}{i!} \right)^{k_i},$$

$$\text{b)}\quad r(n) = \sum_{P_n} n! \prod_{i=1}^{n} (1/k_i!) \left(\frac{g(i)}{i!} \right)^{k_i} s(k,1),$$

$$\text{c)}\quad f(n) = \sum_{P_n} n! \prod_{i=1}^{n} (1/k_i!) \left(\frac{h(i)}{i!} \right)^{k_i}, \text{ and}$$

$$\text{d)}\quad h(n) = \sum_{P_n} n! \prod_{i=1}^{n} (1/k_i!) \left(\frac{f(i)}{i!} \right)^{k_i} s(k,1).$$

If a usable generating function is found for any one of f(n), g(n), h(n) or r(n), then generating functions for the other three values may be derived by using theorems 5.6 and 5.7.

We will next consider a collection of topologies for which a simple enumerating formula is known to exist.

<u>Definition</u> <u>5.8</u> For a given set X of order n, let a closed-open topology be any topology on X which contains all of the complements of its elements relative to X.

Rayburn ([8], lemma 1) has shown that the set of minimal open neighborhoode of any closed-open topology on a finite set X forms a partition of X. Thus the number of closed-open topologies $p(n)$ on a set of order n is equal to the number of partitions of that set.

<u>Remark 5.9</u> (adapted from work by R. Lindahl of Morehead State University) $p(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} p(n-i)$.

<u>Proof</u>: Let $X = \{x_1, x_2, \ldots, x_n\}$. Fix any $x \in X$, and for any given partition of X, let i $(1 \leq i \leq n)$ be the order of the element P of the partition which contains x. Then there are $\binom{n-1}{i-1}$ ways of choosing P and there are $p(n-i)$ ways of choosing the remaining $n-i$ elements of X to form a partition. Hence $p(n)$ is as required.

An outstanding conjecture made by Rayburn in [8] is that $\lim_{n\to\infty} p(n)/f(n) = 0$. In order to prove this conjecture, we will need the following two lemmas:

<u>Lemma 5.10</u> (Rayburn) A finite topology is a closed-open topology if and only if its corresponding topogenous matrix is symmetric with respect to the main diagonal.

<u>Lemma 5.11</u> $p(n) \leq n!$.

<u>Proof</u>: By lemma 5.10, $p(n)$ is also the number of symmetric

nxn topogenous matrices. If an nxn topogenous matrix $A = (a_{ij})$ is symmetric, then $a_{ij} = a_{ji}$ for all i and j, and in particular, $a_{in} = a_{ni}$. Hence if for some $i \varepsilon \{1,2,\ldots,n-1\}$ we have $a_{in} = 1 = a_{in}$ then by corollary 2.6, $a_{n*} = a_{i*}$ and $a_{*n} = a_{*i}$. It follows that there are at most $n-1$ ways of choosing the n^{th} row and column such that for some $i \varepsilon \{1,2,\ldots,n-1\}$, $a_{in}a_{ni} = 1$. Again since A is symmetric, there is exactly one way of choosing $a_{in}a_{ni} = 0$ for all $i = 1,2,\ldots,n-1$ and hence there are at most n possible choices for the n^{th} row and column of a symmetric nxn topogenous matrix. Thus $p(n) \leqq n\cdot p(n-1)$ and by induction we have $p(n) \leqq n!$ as required.

<u>Theorem</u> <u>5.12</u> $\lim\limits_{n\to\infty} p(n)/f(n) = 0.$

<u>Proof</u>: From remark 5.4 we know that for $n > 1$, $f(n) \geqq 2^{n^2/4}$ and by lemma 5.11 we have that $p(n) \leqq n!$. Hence for all $n > 1$, $p(n)/f(n) \leqq n! 2^{-n^2/4}$. Note that $17 < 2^{(17-1)/2}$ and by induction on n, if $n \geqq 17$ then $n < 2^{(n-1)/2}$.

Note also that $16! < \prod_{i=1}^{16} 2^{(i-1)/2}$ and hence by the preceding remark, if $n \geqq 16$ then $n! < \prod_{i=1}^{n} 2^{(n-1)/2} = 2^{n(n-1)/4}$. Thus ,

$$\lim_{n\to\infty} n!\, 2^{-n^2/4} \leqq \lim_{n\to\infty} 2^{n(n-1)/4}\cdot 2^{-n^2/4} = \lim_{n\to\infty} 2^{-n/4} = 0,$$

and hence $\lim\limits_{n\to\infty} p(n)/f(n) = 0$ as required.

As a final theorem in this paper, we will give a non-recursive enumeration of $f(n)$.

<u>Theorem</u> <u>5.13</u> Let $f(n)$ be the number of topologies on a set of order n and for any nxn Boolean matrix A, let :

$$\mu(A) = \begin{cases} 1 & \text{if and only if } A = A^2 \\ 0 & \text{otherwise,} \end{cases}$$

where A^2 is the Boolean matrix product of A with itself. Then:

$$f(n) = \sum \mu(A) \cdot \prod_{i=1}^{n} a_{ii},$$

where the summation runs over all possible nxn Boolean matrices and where a_{ii} is an entry on the main diagonal of the matrix A.

<u>Proof</u>: By theorem 2.2 and lemma 2.3, we have that the number of topologies on a set of order n is equal to the number of nxn reflexive idempotent Boolean matrices. If $A = (a_{ij})$ is any nxn Boolean matrix, then by definition, A is reflexive if and only if the product of the elements of the main diagonal is one, and A is idempotent if and only if $A = A^2$. Hence the theorem follows.

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